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On the discrete time version of the Brussels formalism

O F Bandtlow† and P V Coveney‡

† Cavendish Laboratory, University of Cambridge, Madingley Road, Cambridge CB3 0HE, UK
‡ Schlumberger Cambridge Research, High Cross, Madingley Road, Cambridge CB3 0HG, UK

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Abstract. We develop a discrete time version of the so-called Brussels formalism in non-equilibrium statistical mechanics for continuous endomorphisms of a Banach space. We show that, if the evolution operator U and the projector P are such that PU is a compact operator and the spectral radius of $(I-P)U(I-P)$ is strictly less than the spectral radius of U, then the formalism holds and the evolution operator is quasicompact.

1. Introduction

One of the oldest and most challenging problems in non-equilibrium statistical mechanics, *not to say its raison d'être, is the reconciliation of the irreversible macroscopic* laws governing the behaviour of matter in bulk with the basic time symmetric microscopic equations of motion. A fertile branch in this endeavour was initiated by Ludwig Boltzmann, whose search for the origin of the second law of thermodynamics led him to a characterization of dynamical processes in many-body systems by a kinetic equation, which is now referred to as the Boltzmann equation. Since then numerous such equations have been derived: the Fokker–Planck equation for a Brownian particle, the Vlasov and Balescu–Lenard equations for a plasma, to name but three. Their common feature is that they describe the Markovian dynamics of single particle distributions under certain physical conditions; the Boltzmann equation, for example, gives a correct description only for dilute gases. These conditions are usually formulated in terms of a limiting process for which an appropriately chosen scaling parameter of the system vanishes (see, e.g., [44] and references *therein*).

However, in the past thirty years, a group working in Brussels has developed a formalism which dispenses with this limiting procedure, and provides instead a means of deriving kinetic equations which are valid over some *finite* range of values of the relevant parameter. In essence, the so-called 'Brussels formalism' is based on the construction of an idempotent operator Π, which commutes with the Liouville operator of the system. In other words: Π projects onto a subspace which is invariant under the Liouvillian. It is for this reason that the formalism is also referred to as 'subdynamics'. Moreover, the elements of the subspace can be shown to obey an autonomous evolution equation, which is the desired kinetic equation.

Although this brief summary hardly does justice to the sophistication of the approach it does represent the key aspects of the theory, as it stood in 1975. Further details may be found in the book by Balescu [3] or in the original articles, e.g. [37, 12, 38, 13]. The theory was later generalized to include systems with a time-dependent Liouvillian, in order to describe open systems (see, e.g., [4, 23, 7]).

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In more recent times, the Brussels group, in collaboration with colleagues based in Austin (Texas), has turned much of its attention to the study of so-called 'large Poincaré systems', by which is meant a special class of non-integrable systems characterized by a continuous spectrum (cf [34, 39, 35]). These are introduced by taking the thermodynamic limit of non-integrable systems which contain a finite number of particles. Emphasis has shifted from the derivation of kinetic equations to the derivation of a new spectral representation of the Liouvillian, thereby explicitly admitting the possibility of complex eigenvalues which decay exponentially with time. Nevertheless, the existence of an operator Π, or more generally a set $\{\Pi^{(i)}\}_{i\in I}$ of such operators satisfying the conditions of completeness

$$\sum_{i\in I} \Pi^{(i)} = 1$$

idempotence and orthogonality

$$\Pi^{(i)}\Pi^{(j)} = \delta_{ij}\Pi^{(i)}$$

and commutativity with the Liouville operator

$$L\Pi^{(i)} = \Pi^{(i)}L$$

remains at the heart of the approach.

Another new development was put forward by Hasegawa and Saphir in a series of papers [14–17, 43], who adapted the formalism to the investigation of chaotic mappings. In particular, they were able to derive a generalized spectral decomposition of the Frobenius–Perron operator of the baker's transformation and the Bernoulli map. Their analysis was later extended by Antoniou and Tasaki [1, 2] to the β-adic baker's transformation and the Rényi map.

In this paper we will take up the idea of a discrete time version of the Brussels formalism and focus on conditions for which the formalism holds. For the continuous time scenario Coveney and Penrose [8] have only recently formulated a set of theorems which provide rigorous conditions under which at least a part of the formalism holds in an arbitrary Hilbert space. After a brief definition of the notation used in this paper, we will derive the discrete time analogue of the generalized master equation in a Banach space setting; this leads to the Brussels decomposition of the resolvent of the evolution operator used by Hasegawa and Saphir. We will show that this decomposition rigorously holds if the evolution operator U and the projector P are such that PU is a compact operator. Under the additional assumption that the spectral radius of $(I-P)U(I-P)$ is strictly less than the spectral radius of U we will be able to recover the main features of the Brussels formalism (theorem 2). The class of operators fulfilling these conditions will be shown to be the class of quasicompact operators (theorem 1) and will be studied in some detail in section 5. Finally we give an example of a system for which the evolution operator is quasicompact.

2. Notation

Throughout the present paper, z denotes a complex number and X a non-zero complex Banach space. We use the notation $\mathcal{L}(X)$ for the Banach algebra of bounded linear operators on X and $\mathcal{K}(X)$ for the closed two-sided ideal of compact operators in $\mathcal{L}(X)$. For $T \in \mathcal{L}(X)$, the symbols $\varrho(T)$ and $\sigma(T)$ will be used for the resolvent set and the spectrum of T respectively; $r(T)$ denotes the spectral radius of T and $A(T) := \{z : |z| > r(T)\}$ the annulus of convergence of the von Neumann series of $(z-T)^{-1}$. Finally, we write $\mathcal{N}(T)$ for the kernel and $\mathcal{R}(T)$ for the range of T.

3. The discrete time master equation

One of the earliest attempts to generalize Boltzmann's kinetic equation to arbitrary systems was made by Pauli [33] who derived a master equation for the time evolution of the probability distribution of a quantum system by assuming that this was driven by the random steps in a Markov process, a hypothesis which is not in general consistent with the Liouville equation. Pauli's approach was later improved by van Hove [21, 22], while Prigogine and his collaborators [40] arrived at an exact master equation for an arbitrary system. Similar equations were derived by Nakajima [32], Zwanzig [45], and Montroll [29]. Their equivalence was shown by Zwanzig [46].

To keep the discussion general we take a Banach space X as a state space and $U \in \mathcal{L}(X)$ as the generator of the dynamical semigroup $\{U^n\}_{n\in\mathbb{N}}$.

The derivation of the discrete time master equation starts by introducing a pair of projectors P and Q with $P, Q \in \mathcal{L}(X)$ and $Q = I - P$ into the difference equation of the dynamical semigroup

$$U_{n+1} = UU_n$$

with the initial condition

$$U_0 = I$$

which leads to the following pair of equations:

$$PU_{n+1} = PUPU_n + PUQU_n \tag{1}$$

$$QU_{n+1} = QUPU_n + QUQU_n\,. \tag{2}$$

The Brussels school refers to the P and Q subspaces as the 'vacuum' and the 'correlations', since in the original formalism P was meant to project on the diagonal part of the density matrix of the system. It is important to state that there is nothing explicitly required of the dimensionality of these projectors which, according to this school, may be either finite or infinite.

Although this pair of operator equations can be solved by iteration, it is much easier to use z-transform techniques, which are the discrete time analogue of Laplace transforms (the appendix should be consulted for more details). It is not difficult to see that all the terms occurring in (1) and (2) are of geometric order owing to the submultiplicativity of the operator norm in a Banach space. We have, for example,

$$\|PUPU_n\| \leqslant \|P\|^2\,\|U\|^{n+1}\,.$$

We may, therefore, apply a z-transform and get

$$z(P\mathcal{U}(z) - PU_0) = PUP\mathcal{U}(z) + PUQ\mathcal{U}(z) \tag{3}$$

$$z(Q\mathcal{U}(z) - PU_0) = QUP\mathcal{U}(z) + QUQ\mathcal{U}(z) \tag{4}$$

where we used the shifting theorem (see the appendix) and the definition

$$\mathcal{U}(z) := \mathcal{Z}[U_n] = \frac{z}{z-U}\,.$$

For $z \in A(QUQ)$ we can solve for $Q\mathcal{U}(z)$ in (4)

$$Q\mathcal{U}(z) = \frac{z}{z-QUQ}Q + \frac{1}{z-QUQ}QUP\mathcal{U}(z) \tag{5}$$

where we used $U_0 = 1$. Inserting this into (3) yields

$$z(P\mathcal{U}(z) - P) = PUP\mathcal{U}(z) + PUQ\frac{z}{z - QUQ} + PUQ\frac{1}{z - QUQ}QUP\mathcal{U}(z)\,. \tag{6}$$

The Brussels school coined suggestive names for the continuous time analogues of the operators in (5) and (6), which we shall also adopt: the 'collision operator'

$$\tilde{\psi}(z) := PUQ\frac{1}{z - QUQ}QUP \tag{7}$$

the 'destruction operator'

$$\mathcal{D}(z) := PUQ\frac{1}{z - QUQ} \tag{8}$$

the 'creation operator'

$$\mathcal{C}(z) := \frac{1}{z - QUQ}QUP \tag{9}$$

and the 'reduced resolvent'

$$\mathcal{S}(z) := Q\frac{1}{z - QUQ}Q\,. \tag{10}$$

All of them are $\mathcal{L}(X)$ -valued functions holomorphic in $A(QUQ)$. For the sake of completeness we list their n-domain representations, that is their images under an inverse z-transform

$$\psi_n := \mathcal{Z}_n^{-1}[\tilde{\psi}(z)] = \begin{cases} PUQ(QUQ)^{n-1}QUP & n \geqslant 1 \\ 0 & n = 0 \end{cases} \tag{11}$$

$$D_n := \mathcal{Z}_n^{-1}[\mathcal{D}(z)] = \begin{cases} PUQ(QUQ)^{n-1} & n \geqslant 1 \\ 0 & n = 0 \end{cases} \tag{12}$$

$$C_n := \mathcal{Z}_n^{-1}[\mathcal{C}(z)] = \begin{cases} (QUQ)^{n-1}QUP & n \geqslant 1 \\ 0 & n = 0 \end{cases} \tag{13}$$

$$S_n := \mathcal{Z}_n^{-1}[\mathcal{S}(z)] = \begin{cases} Q(QUQ)^{n-1}Q & n \geqslant 1 \\ 0 & n = 0\,. \end{cases} \tag{14}$$

The desired master equation may now be obtained from (6) *quā* inverse z-transform

$$PU_{n+1} = PUPU_n + D_{n+1}Q + \psi_n * PU_n \tag{15}$$

where '$*$' denotes the convolution of two sequences

$$\psi_n * PU_n := \sum_{i=0}^{n} \psi_i PU_{n-i}\,.$$

Equation (15) is an operator identity which, acting on an initial state $f_0 \in X$, provides a relation for the P-component of the iterates $f_n := U_n f_0$ of f_0

$$Pf_{n+1} = PUPf_n + D_{n+1}Qf_0 + \psi_n * Pf_n\,. \tag{16}$$

This is a discrete time version of the generalized master equation.

We have thus arrived at an *exact* equation for the evolution of the reduced densities Pf_n. The second term on the right-hand side of (16) describes the influence of initial data about Qf_0 at time $n = 0$ on the subsequent time evolution of the system. If Q is appropriately chosen it can be assumed that its effect should disappear in the long time (large n) limit. Equation (16) is, however, non-Markovian, due to the summation present in the third or collision term. Thus, to pass from this equation to a Markovian equation, we need to restrict the influence of ψ_n for large n. We will turn to this problem in section 6.

4. The Brussels class

In order to understand the reasoning behind the Brussels formalism, we need to derive an expression for $\mathcal{U}(z)$ in terms of the operators previously introduced. The z-transformed version of the master equation (15) is

$$z(P\mathcal{U}(z) - P) = PUP\mathcal{U}(z) + z\mathcal{D}(z) + \tilde{\psi}(z)P\mathcal{U}(z).$$

We can formally solve for $P\mathcal{U}(z)$

$$P\mathcal{U}(z) = \frac{z}{z - PUP - \tilde{\psi}(z)}[P + \mathcal{D}(z)]$$

which, added to the equation for $Q\mathcal{U}(z)$ (5)

$$Q\mathcal{U}(z) = z\mathcal{S}(z) + \mathcal{C}(z)P\mathcal{U}(z)$$

yields the 'Brussels decomposition' of $\mathcal{U}(z)$, namely

$$\begin{aligned}\mathcal{U}(z) &= [P + \mathcal{C}(z)]P\mathcal{U}(z) + z\mathcal{S}(z)\\ &= [P + \mathcal{C}(z)]\frac{z}{z - PUP - \tilde{\psi}(z)}[P + \mathcal{D}(z)] + z\mathcal{S}(z). \end{aligned} \tag{17}$$

For a justification of the manipulations involved so far the existence of $[z - PUP - \tilde{\psi}(z)]^{-1}$ has to be ensured. This can be done by imposing the condition that P and U are such that PU is compact. Note that $\dim PX < \infty$ is a sufficient but not necessary condition for PU to be compact.

Proposition 1. Let U, P, and Q be defined as above, with $PU \in \mathcal{K}(X)$. Then $z/[z - PUP - \tilde{\psi}(z)]$ is

(i) meromorphic in the annulus $A(QUQ)$ with only a finite number of poles z_i, $i \in I$
(ii) holomorphic at infinity, i.e. $(1/z)/[1/z - PUP - \tilde{\psi}(1/z)]$ is holomorphic at 0.

Proof. For $z \in A(QUQ)$ the operator $\frac{1}{z}(PUP - \tilde{\psi}(z))$ is holomorphic. Since the product of a compact operator and a bounded operator is compact $\frac{1}{z}(PUP - \tilde{\psi}(z))$ is also compact due to PU being compact. Furthermore $1 - \frac{1}{z}(PUP - \tilde{\psi}(z))$ is invertible for z large enough. This is easily seen by taking into account that

$$\|\tilde{\psi}(z)\| \leqslant (\|P\|\;\|U\|\;\|Q\|)^2\,(|z| - \|QUQ\|)^{-1}$$

becomes arbitrary small for z large, and hence

$$\left\|\tfrac{1}{z}(PUP - \tilde{\psi}(z))\right\| < 1$$

for z large enough. The first assertion now follows from the analytic Fredholm theorem (see [41, theorem VI.14] and [10, theorem VII.11]). For the proof of the second part let $z \in \{z : |z| < 1/r(QUQ)\}$ for $r(QUQ) \neq 0$ or z arbitrary if $r(QUQ) = 0$. Then

$$z\tilde{\psi}(1/z) = \sum_{n=0}^{\infty} z^{n+2} PUQ(QUQ)^n QUP$$

hence $[1 - zPUP - z\tilde{\psi}(1/z)]^{-1}$ holomorphic at 0. Using the same expansion it is possible to show that

$$\lim_{z\to 0} \left\| zPUP - z\tilde{\psi}(1/z) \right\| = 0$$

and therefore that $[1 - zPUP - z\tilde{\psi}(1/z)]^{-1}$ is invertible at $z = 0$. This completes the proof. □

Remark. The same arguments may be used to prove a slightly extended version of the proposition, in which the annulus $A(QUQ)$ is replaced by an arbitrary connected open subset of $\varrho(QUQ)$.

This proposition justifies the application of an inverse z-transform to equation (17) thus yielding a new expression for U_n

$$U_n = \frac{1}{2\pi i} \oint_C z^n \left\{ (P + \mathcal{C}(z)) \frac{1}{z - PUP - \tilde{\psi}(z)} (P + \mathcal{D}(z)) + \mathcal{S}(z) \right\} \mathrm{d}z\,.$$

The contour C has to enclose the poles $z_i, i \in I$ as well as $\sigma(QUQ)$. As the integrand is meromorphic in $A(QUQ)$ we can deform C such as to separate the contributions from the poles resulting in a splitting of the integral

$$U_n = \Sigma_n^{(1)} + \Sigma_n^{(2)} + \cdots + \Sigma_n^{(p)} + \hat{\Sigma}_n \tag{18}$$

where $p = \operatorname{card} I$ is the number of poles of the integrand in $A(QUQ)$ and for $1 \leqslant i \leqslant p$

$$\begin{aligned}
\Sigma_n^{(i)} &:= \frac{1}{2\pi i} \oint_{C_i} z^n \left\{ [P + \mathcal{C}(z)] \frac{1}{z - PUP - \tilde{\psi}(z)} [P + \mathcal{D}(z)] \right\} \mathrm{d}z \\
\hat{\Sigma}_n &:= \frac{1}{2\pi i} \oint_{C'} z^n \left\{ (P + \mathcal{C}(z)) \frac{1}{z - PUP - \tilde{\psi}(z)} (P + \mathcal{D}(z)) + \mathcal{S}(z) \right\} \mathrm{d}z\,.
\end{aligned} \tag{19}$$

The new contours C_i enclose the poles z_i only and C' is a circle around the origin with radius $r(QUQ) + \epsilon$ with $\epsilon > 0$ small enough.

Note that the contribution from $\mathcal{S}(z)$ in (19) vanishes, since $\mathcal{S}(z)$ is holomorphic in $A(QUQ)$. The operators $\Sigma_n^{(i)}$ are the so-called 'asymptotic evolution operators', which play an important *rôle* in the Brussels approach; we shall study them further in section 6. For the moment we only remark that these operators are supposed to describe the dominant long time behaviour of U_n.

In order to obtain the splitting of the evolution in (18) the existence of a pole in $A(QUQ)$ needs to be ensured. It obviously suffices to require that

$$r(QUQ) < r(U)$$

which is a mathematical formulation of the hypothesis of rapid decay of correlations frequently assumed in derivations of Markovian kinetic equations.

We now cast these results into the following definition.

Definition. Let X be a Banach space. An operator $U \in \mathcal{L}(X)$ is said to belong to the *Brussels class* $\mathcal{Q}'(X)$ of X if there is a projector $Q \in \mathcal{L}(X)$ such that $(I-Q)U$ is compact and $r(QUQ) < r(U)$.

An explicit characterization of the Brussels class is given in the next section.

5. Quasicompact operators and the Brussels class

To give a characterization of the Brussels class put forward in the previous section we introduce a class of operators called quasicompact operators.

Definition. A bounded operator U is said to be *quasicompact* if there is a $k \in \mathbb{N}$ and a compact operator K such that

$$\left\|U^k - K\right\| < r(U)^k .$$

The set of all quasicompact operators on X will be denoted by $\mathcal{Q}(X)$.

We begin by proving some elementary properties of quasicompact operators

Lemma 1. Let $U \in \mathcal{L}(X)$ be an operator such that U^k is compact for some $k \in \mathbb{N}_0$. Then U is quasicompact if and only if it is not quasinilpotent, i.e. $r(U) \neq 0$.

Proof. The proof follows directly from $\left\|U^k - U^k\right\| = 0 < r(U)$. □

For a new characterization of $\mathcal{Q}(X)$ we need the following definition.

Definition. Let $T \in \mathcal{L}(X)$. Define

$$\kappa(T) := \inf\{\|T-K\| : K \in \mathcal{K}(X)\} .$$

We can now formulate the following lemma.

Lemma 2. An operator $U \in \mathcal{L}(X)$ is quasicompact if and only if

$$\lim_{n\to\infty} \kappa(U^n)^{1/n} < r(U) .$$

Proof. The 'if' part is trivial. For the 'only if' part we show that

$$\kappa(U^{m+n}) \leqslant \kappa(U^m)\kappa(U^n) \tag{20}$$

because then the sequence $\kappa(U^n)^{1/n}$ converges to its greatest lower bound (by [36, section I, problem 98]) and the assertion follows. To prove (20) observe that for $K_1, K_2 \in \mathcal{K}(X)$

$$\begin{aligned}\kappa(U^{m+n}) &\leqslant \left\|U^{m+n} - (U^m K_2 + K_1 U^n - K_1 K_2)\right\| \\ &\leqslant \left\|U^m - K_1\right\| \left\|U^n - K_2\right\| .\end{aligned}$$ □

The sum and the product of two quasicompact operators need not be quasicompact, as the following example shows.

Example. Let $X = l^2$. Define

$$x = (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, \ldots)$$

$$Sx = (x_1, 0, 0, x_3, 0, x_5, 0, x_7, \ldots)$$

$$Tx = (0, x_2, x_4, 0, x_6, 0, x_8, 0, \ldots)$$

$$STx = (0, 0, 0, x_4, 0, x_6, 0, x_8, \ldots)$$

$$(S + T)x = (x_1, x_2, x_4, x_3, x_6, x_5, x_8, x_7, \ldots).$$

Observe that $S, T \in \mathcal{L}(X)$ and that for $n \in \mathbb{N}_0$

$$S^{n+2}x = (x_1, 0, 0, \ldots)$$

$$T^{n+2}x = (0, x_2, 0, \ldots)$$

$$(ST)^n = ST$$

$$(S+T)^{2n} = I$$

$$(S+T)^{2n+1} = S + T.$$

Using the spectral radius formula we get

$$r(S) = \lim_{n\to\infty} \|S^n\|^{1/n} = \lim_{n\to\infty} \|S^2\|^{1/n} = 1$$

and with a similar argument

$$r(T) = r(ST) = r(S+T) = 1.$$

Now, since $S^2, T^2 \in \mathcal{K}(X)$ and S and T are not quasinilpotent, S and T are quasicompact by lemma 1. However, ST is a projector with an infinite dimensional range, and therefore $ST \notin \mathcal{K}(X)$. Since $\mathcal{K}(X)$ is closed in $\mathcal{L}(X)$ it follows that

$$\kappa((ST)^n)^{1/n} \to 1 \qquad \text{as } n \to \infty.$$

Therefore ST is not quasicompact by lemma 1. Furthermore $(S+T)^{2n} = I \notin \mathcal{K}(X)$ and again it follows that $S + T$ is not quasicompact.

Unlike $\mathcal{K}(X)$ the class of quasicompact operators $\mathcal{Q}(X)$ is not a subspace of $\mathcal{L}(X)$ and not an ideal of $\mathcal{L}(X)$ in general. Nevertheless the following is true.

Proposition 2. U is quasicompact if and only if its adjoint U^* is quasicompact.

Proof. Since an endomorphism K of a Banach space is compact if and only if its adjoint is compact (see, e.g., [19, propositions 42.2 and 42.3]) we get

$$\|(U^*)^k - K^*\| = \|U^k - K\| < r(U) = r(U^*). \qquad \square$$

$\mathcal{Q}(X)$ is not closed in $\mathcal{L}(X)$. We show this by giving an example of a sequence of quasicompact operators converging uniformly to an operator which is not quasicompact.

Example. Let $X = l^2$ and define $\{S_n\}_{n\in\mathbb{N}} \subset \mathcal{L}(X)$ through

$$S_n x = ((1/2)^n x_1, 0, 0, x_3, 0, x_5, \ldots).$$

Then for $j \geqslant 2$

$$S_n^j x = ((1/2)^{nj} x_1, 0, 0, \ldots) \in \mathcal{K}(X)$$

and since $r(S_n) = (1/2)^n$, every S_n is quasicompact. However, $\lim_{n\to\infty} S_n = 0$, which is not quasicompact.

In order to see why quasicompact operators appear in this context we recall Browder's definition of the *essential spectrum* $\sigma_{\mathrm{ess}}(T)$ of an operator $T \in \mathcal{L}(X)$ [6].

An element $z \in \sigma(T)$ is said to belong to $\sigma_{\mathrm{ess}}(T)$ if one or more of the following is true:

(i) $\mathcal{R}(z - T)$ is not closed in X
(ii) z is a limit point of $\sigma(T)$
(iii) $\bigcup_{r=1}^{\infty} \mathcal{N}(z - T)^r$ is infinite dimensional.

By analogy with the spectral radius of T the *essential spectral radius* $r_{\mathrm{ess}}(T)$ is defined to be

$$r_{\mathrm{ess}}(T) = \sup\{|z| : z \in \sigma_{\mathrm{ess}}(T)\}\,.$$

There are various other definitions of the essential spectrum in the literature, and in general they are not equivalent. Fortunately for any of the standard definitions the essential spectral radius is the same (see [11, I.4]) and is given by the Nussbaum formula [31]

$$r_{\mathrm{ess}}(T) = \lim_{n\to\infty} \kappa(T^n)^{1/n}\,. \tag{21}$$

Using the above results, we are now able to prove the following lemma.

Lemma 3. Let U be a quasicompact operator. Then for every $0 < \epsilon \leqslant (r(U) - r_{\mathrm{ess}}(U))$ the set $\sigma_\epsilon(U) := \sigma(U) \cap \{z : |z| \geqslant r_{\mathrm{ess}}(U) + \epsilon\}$ is not empty and consists of a finite set of eigenvalues with finite multiplicity.

Proof. For U quasicompact we conclude from lemma 2 and equation (21) that $r_{\mathrm{ess}}(U) < r(U)$. The set $\sigma_\epsilon(U)$ is not empty, since at least one point of $\sigma(U)$ lies on the circle $\{z : |z| = r(U)\}$. For every $z \in \sigma_\epsilon(U)$, the range of $z - U$ is dense in X but $z - U$ is not invertible, hence $z - U$ is not injective and z is an eigenvalue with finite multiplicity by the definition of $\sigma_\epsilon(U)$. Finally, since $\sigma_\epsilon(U)$ is compact and contains no limit points, it can only consist of a finite number of elements. □

Using an argument by Keller [27] we are now able to prove the following theorem which constitutes a complete description of the Brussels class.

Theorem 1. A bounded operator on a Banach space belongs to the Brussels class if and only if it is quasicompact

$$\mathcal{Q}'(X) = \mathcal{Q}(X)\,.$$

Proof. '⇒' (See [27, proposition 2.2]) Let U belong to the Brussels class. Then there is a projector $Q \in \mathcal{L}(X)$ such that $r(QUQ) < r(U)$ and $P := I - Q$ with $PU \in \mathcal{K}(X)$. We show by induction on n that

$$U^n - (QUQ)^{n-1}U \text{ is compact for all } n \in \mathbb{N}\,. \tag{22}$$

For $n = 1$ this is trivial. Assuming that equation (22) holds for n, then

$$\begin{aligned} U^{n+1} - (QUQ)^n U &= PUU^n + QUU^n - QU(QUQ)^{n-1}U + QUP(QUQ)^{n-1}U \\ &= PUU^n + QUP(QUQ)^{n-1}U + QU(U^n - (QUQ)^{n-1}U)\,. \end{aligned}$$

The first term in this sum is compact since PU is compact. The second term is only different from 0 for $n = 1$, in which case its compactness follows from that of PU, whereas the last term is compact by the induction assumption. That U is quasicompact now follows from

$$\lim_{n\to\infty} \|(QUQ)^{n-1}U)\|)^{1/n} = r(QUQ) < r(U)\,.$$

'$\Leftarrow$' Let $\epsilon > 0$. Since U is quasicompact P can be chosen to be the projector onto the eigenspaces of the eigenvalues of U in $\sigma_\epsilon(U)$, by lemma 3 P is a finite rank operator, which implies that PU is compact. The inequality $r(QUQ) < r(U)$ follows from the fact that $QUQ = QU = UQ$ has no eigenvalues in $\sigma_\epsilon(U)$. □

6. Subdynamics

Let us return to equation (18). We show that this splitting of the evolution operator gives rise to independent 'subdynamics' in the following sense.

Theorem 2. Let U belong to the Brussels class and let $z_i, i \in \{1, \dots, p\}$ denote the poles of $[z - PUP - \tilde{\psi}(z)]^{-1}$ in $A(QUQ)$. Then $p \geqslant 1$ and there are $p + 1$ bounded projectors $\Pi^{(i)}, i \in \{0, \dots, p\}$ with

$$\sum_{i=0}^{p} \Pi^{(i)} = 1 \tag{23}$$

$$\Pi^{(i)}\Pi^{(j)} = \delta_{ij}\Pi^{(i)} \text{ for } i, j \in \{0, \dots, p\} \tag{24}$$

$$U\Pi^{(i)} = \Pi^{(i)}U \text{ for } i \in \{0, \dots, p\}. \tag{25}$$

Moreover the asymptotic evolution operators $\Sigma_n^{(i)}$ can be written

$$\Sigma_0^{(i)} = \Pi^{(i)} \tag{26}$$

$$\Sigma_n^{(i)} = z_i^n \Pi^{(i)} + \sum_{l=1}^{n} \binom{n}{l} z_i^{n-l} \Delta_i^l \tag{27}$$

where

$$\Delta_i := \frac{1}{2\pi i} \oint_{C_i} (z - z_i) \frac{1}{z - U} \, \mathrm{d}z$$

and C_i is a contour enclosing the pole z_i only. The remainder operator $\hat{\Sigma}_n$ obeys

$$\left\| \hat{\Sigma}_n \right\| \leqslant K a^n \tag{28}$$

for some constant K and $r(QUQ) < a < \min_{i \in \{1,\dots,p\}}\{|z_i|\}$.

Proof. This proceeds directly. Let $0 < \epsilon < \min_{i \in \{1,\dots,p\}}\{|z_i|\} - r(QUQ)$ and observe that the Brussels decomposition of the resolvent (17) is valid for a connected subset of the complex z-plain, that is for $\{z : |z| \geqslant r(QUQ) + \epsilon\}$ except for a finite number of points, due to Proposition 1 and U being quasicompact. We may therefore replace the integrand in (19) by $z^n/(z - U)$ and use the following Laurent series expansion:

$$\frac{1}{z - U} = \sum_{i=1}^{p} \left[\frac{\Pi^{(i)}}{z - z_i} + \sum_{l=1}^{\infty} \frac{\Delta_i^l}{(z - z_i)^{l+1}} \right] + \frac{1}{z - U} \Pi^{(0)}$$

where $\Pi^{(i)}$ and Δ_i are the eigenprojection and the eigennilpotent associated with z_i, and $\frac{1}{z-U}\Pi^{(0)}$ is holomorphic in $z_i, i \in \{1, \dots, p\}$ with

$$\Pi^{(0)} := 1 - \left(\Pi^{(1)} + \dots + \Pi^{(p)} \right).$$

This is a standard result and may be found in [25, III.6.5]. Now trivially (23), (24) and (25) hold. Finally (26) and (27) follow from a simple integration

$$\frac{1}{2\pi i}\oint_{C_i}\frac{z^n}{(z-z_j)^{l+1}}\,\mathrm{d}z = \begin{cases} \delta_{ij}\binom{n}{l} z_j^{n-l} & \text{for } l \leqslant n \\ 0 & \text{otherwise} \end{cases}$$

while (28) is a consequence of

$$\hat{\Sigma}_n = \frac{1}{2\pi i}\oint_{C'}\frac{z^n}{z-U}\Pi^{(0)}\,\mathrm{d}z$$

and the fact that $\frac{1}{z-U}\Pi^{(0)}$ is analytic in $\{z : |z| \geqslant r(QUQ)+\epsilon\}$. □

We have recovered the basic features of subdynamics as defined by the Brussels school. The temporal evolution of the system may be separated into independently evolving parts by virtue of the projectors $\Pi^{(i)}$. The long time behaviour in the subspace $\Pi^{(i)}X$ is governed by (27). Note that since

$$\Delta_i^n = 0 \text{ for } n \geqslant \nu_i$$

where ν_i is the algebraic multiplicity of z_i , i.e. the dimension of $\Pi^{(i)}X$, the evolution of a probability density f_0 at time $n = 0$ entirely lying in $\Pi^{(i)}X$ for n large (i.e. for $n \geqslant \nu_i$) is given by

$$f_n = z_i^n f_0 + n z_i^{n-1}\Delta_i f_0 + \cdots + \frac{n!}{(n-\nu+1)!(\nu-1)!} z_i^{n-\nu+1}\Delta_i^{\nu-1} f_0 \,.$$

Thus, for a mode with non-vanishing eigennilpotent, i.e. for eigenvectors belonging to a an eigenvalue which is not simple, we get a coupling of the generalized eigenvectors. The decay, however, will still be exponential, since $\|f_n\|$ is dominated by $(|z_i|+\epsilon)^n$ for every $\epsilon > 0$.

7. The β-transformation

We now give an example of a dynamical system for which the associated Frobenius–Perron operator (see for example [28]) on a suitably chosen Banach space is quasicompact. More explicitly, we shall study the following map:

$$T : [0,1] \to [0,1]$$

$$Tx = \beta x \bmod 1 \qquad \beta \in \mathbb{N}^+ \,.$$

This map is usually referred to as the 'β-transformation' or the 'β-adic Rényi map'. It has been extensively studied throughout the last 30 years and is nowadays considered to be the simplest example of a chaotic system. Only recently a generalized spectral decomposition of the associated Frobenius–Perron operator for $\beta = 2$ was obtained by Hasegawa and Saphir [43, 17] and for $\beta \in \mathbb{N}^+$ by Antoniou and Tasaki [1].

The Frobenius–Perron operator U of the dynamical system (T, λ), where λ denotes Lebesgue measure, can easily be calculated [28]

$$U : L^1(\lambda) \to L^1(\lambda)$$

$$Uf(x) = \beta^{-1}\sum_{i=0}^{\beta-1} f(\beta^{-1}(x+i)) \,.$$

Keller [26] showed that the L^1-spectrum of the Frobenius–Perron operator of a non-invertible transformation is the closed unit disk, hence we cannot expect U to be quasicompact on $L^1(\lambda)$. Nevertheless, the operator U turns out to be quasicompact when its domain is restricted to certain dense subspaces of $L^1(\lambda)$, as we shall now prove .

To this end, recall that for $m \in \mathbb{N}$, $\alpha \in (0.1]$ the space $C^{m,\alpha}$ of all complex-valued m-times differentiable functions on $[0.1]$, the mth derivative of which is Hölder-continuous with exponent α, is dense in $L^1(\lambda)$ and becomes a complex Banach space when furnished with the norm

$$\|f\|_{m,\alpha} = |f|_m + \|f^{(m)}\|_\alpha$$

where

$$|f|_m := \max_{0\leqslant j\leqslant m} \sup_{x\in[0,1]} |f^{(j)}|$$

and

$$\|f\|_\alpha := \sup_{\substack{x,y\in[0,1]\\ x\neq y}} \frac{|f(x)-f(y)|}{|x-y|^\alpha}\,.$$

We shall see in the proof of proposition 3 that U is a bounded linear operator on $(C^{m,\alpha}, \|.\|_{m,\alpha})$. In order to show that U is even quasicompact we need the following version of the Ionescu-Tulcea Marinescu ergodic theorem given by Hennion [18]

Theorem 3 (Hennion, 1993). Let $(X, \|.\|)$ be a Banach space and U a bounded linear operator on $(X, \|.\|)$. If there is a norm $|.|$ on X, such that

(i) $U : (X, \|.\|) \to (X, |.|)$ is compact
(ii) for every $n \in \mathbb{N}$, there are positive reals R_n, r_n, such that $\liminf_{n\to\infty}(r_n)^{1/n} =: r < r(T)$

$$\left\|U^n f\right\| \leqslant R_n\,|f| + r_n\,\|f\| \quad \text{for all } f \in X$$

then U is quasicompact and $r_{\mathrm{ess}}(U) \leqslant r$.

We are now able to prove the main result of this section.

Proposition 3. Let U be the Frobenius–Perron operator of the β-transformation (T, λ), then for $m \in \mathbb{N}, \alpha \in (0, 1]$ the operator U is a quasicompact endomorphism of $(C^{m,\alpha}, \|.\|_{m,\alpha})$ and $r_{\mathrm{ess}}(U) \leqslant \beta^{-(m+\alpha)}$.

Proof. Fix $m \in \mathbb{N}$ and $\alpha \in (0, 1]$. Let $f \in C^{m,\alpha}$, then

$$Uf(x) = \beta^{-1}\sum_{i=0}^{\beta-1} f(\beta^{-1}(x+i))$$

is m-times continuously differentiable and for $0 \leqslant j \leqslant m$

$$(Uf)^{(j)}(x) = \beta^{-(j+1)}\sum_{i=0}^{\beta-1} f^{(j)}(\beta^{-1}(x+i))\,.$$

Furthermore, we have the following inequalities, the proof of which we shall supply later

$$|Uf|_m \leqslant |f|_m \tag{29}$$

$$\|(Uf)^{(m)}\|_\alpha \leqslant \beta^{-(m+\alpha)}\|f^{(m)}\|_\alpha\,. \tag{30}$$

Now, (30) implies that $Uf \in C^{m,\alpha}$. Moreover, combining (29) and (30) yields

$$\begin{aligned} \|Uf\|_{m,\alpha} &= |Uf|_m + \|(Uf)^{(m)}\|_\alpha \\ &\leqslant |f|_m + \beta^{-(m+\alpha)} \|f^{(m)}\|_\alpha \qquad (31) \\ &\leqslant \|f\|_{m,\alpha} \, . \qquad (32) \end{aligned}$$

Hence $U \in \mathcal{L}(C^{m,\alpha})$ with the operator norm of U obeying $\|U\| \leqslant 1$. Since $U1 = 1$, we have $r(U) = 1$.

Equation (31) also implies

$$\|Uf\|_{m,\alpha} \leqslant |f|_m + \beta^{-(m+\alpha)} \|f\|_{m,\alpha}$$

which can be iterated to give for $n \in \mathbb{N}^+$

$$\left\|U^n f\right\|_{m,\alpha} \leqslant \sum_{i=0}^{n-1} \beta^{-i(m+\alpha)} |f|_m + \beta^{-n(m+\alpha)} \|f\|_{m,\alpha} \, .$$

Clearly, $|.|_m$ is a norm on $C^{m,\alpha}$; since the natural embedding of $C^{m,\alpha}$ in C^m is compact (see for example [11, theorem V.1.1]), every $\|.\|_{m,\alpha}$-bounded set in $C^{m,\alpha}$ is $|.|_m$-relatively compact and therefore

$$U : (C^{m,\alpha}, \|.\|_{m,\alpha}) \to (C^{m,\alpha}, |.|_{m,})$$

is compact. The assertion of the proposition now follows from theorem 3: U is quasicompact and $r_{\text{ess}}(U) \leqslant \beta^{-(m+\alpha)}$. We only need to prove (29) and (30). Inequality (29) follows from (7), which yields

$$\begin{aligned} \sup_{x\in[0,1]} |(Uf)^{(j)}(x)| &= \beta^{-(j+1)} \sum_{i=0}^{\beta-1} \sup_{x\in[0,1]} |f^{(j)}(\beta^{-1}(x+i))| \\ &\leqslant \beta^{-m} \sup_{x\in[0,1]} |f^{(j)}(x)| \end{aligned}$$

for $0 \leqslant j \leqslant m$, and therefore $|Uf|_m \leqslant |f|_m$. Finally we have the following estimates:

$$\begin{aligned} \|(Uf)^{(m)}\|_\alpha &= \sup_{\substack{x,y\in[0,1]\\ x\neq y}} \beta^{-(m+1)} \frac{|\sum_{i=0}^{\beta-1} (f^{(m)}(\beta^{-1}(x+i)) - f^{(m)}(\beta^{-1}(y+i)))|}{|x-y|^\alpha} \\ &\leqslant \beta^{-(m+1)} \sum_{i=0}^{\beta-1} \sup_{\substack{x,y\in[0,1]\\ x\neq y}} \frac{|f^{(m)}(\beta^{-1}(x+i)) - f^{(m)}(\beta^{-1}(y+i))|}{|x-y|^\alpha} \\ &= \beta^{-(m+1)} \sum_{i=0}^{\beta-1} \sup_{\substack{x,y\in[\beta^{-1}i,\beta^{-1}(i+1)]\\ x\neq y}} \frac{|f^{(m)}(x) - f^{(m)}(y)|}{\beta^\alpha |x-y|^\alpha} \\ &\leqslant \beta^{-(m+\alpha)} \|f^{(m)}\|_\alpha \end{aligned}$$

which proves inequality (30). □

8. Discussion

Our work shows that the possibility of an analytic continuation of the Brussels operators inside the spectrum of the evolution operator U, together with a restriction to compact projectors P (such that $PU \in \mathcal{K}(X)$), guarantees the existence of independent subdynamics for U. This result is somewhat similar to the investigation of Coveney and Penrose [8], who have shown that in the continuous time scenario the existence of an isolated pole of the resolvent of U below the real axis is ensured whenever the time-domain collision operator is bounded above in norm by an exponentially decaying function of time and the projector P is a finite range operator. The connection with our result is seen by taking into account that for our definition of the Brussels class the discrete time-domain collision operator is of geometric order (the discrete time analogue of exponentially bounded) with the least such bound being less than $r(U)$.

The requirement imposed by our analysis for the existence of the discrete time Brussels formalism in statistical mechanics, namely that the evolution operators in question must be quasicompact, is less restrictive than it might appear to be at first sight. Convergence results, such as central limit theorems or exponential decay of correlations obtained via spectral properties of the Frobenius–Perron operator are usually linked to finding suitable restrictions of the domain of the Frobenius–Perron operator on which it is quasicompact. For example, in their studies of the ergodic properties of piecewise monotonic transformations of the interval Hofbauer and Keller [20], Rychlik [42], and Keller [26] make use of the fact that the induced Frobenius–Perron operator, the spectrum of which is the whole unit disk when considered as an endomorphism of L^1, is quasicompact on the space of functions of bounded variation. This situation is reminiscent of the recently developed rigged Hilbert space approach to the Brussels formalism, wherein a spectral representation of indecomposable operators of a Hilbert space can be obtained for a suitable restriction of the domain of the relevant operator [14, 15, 1, 2]. The analysis carried out in the present paper also shares various features in common with the measure-theoretic approach to the selection of 'canonical' non-equilibrium ensembles recently developed by Coveney and Penrose [9]. We hope to return in the future with a more detailed examination of these particular relationships.

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Appendix. The $\mathcal{Z}$-transform

The material covered here is an easy generalization of the standard methods (see, e.g., [24, 30]).

Let X be a complex Banach space. A sequence $\{T_n\}_{n\in\mathbb{N}}$ of bounded operators $T_n \in \mathcal{L}(X)$ is said to be of *geometric order*, if there exist positive reals, A and a, and an integer n_0, such that for all $n \geqslant n_0$

$$\|T_n\| \leqslant Aa^n . \tag{A1}$$

Then the *z-transform* of $\{T_n\}_{n\in\mathbb{N}}$ is defined by

$$\mathcal{Z}[T_n] := \mathcal{T}(z) := \sum_{n=0}^{\infty} T_n z^{-n} .$$

Theorem 4 (Properties). Let $\{T_n\}_{n\in\mathbb{N}}$ be a sequence of geometric order with constant a as in (A1). The z-transform $\mathcal{Z}[T_n]$ is unique and holomorphic in the extended annulus $\{z : |z| > a\} \cup \{\infty\}$.

Proof. The assertion follows from the fact that $\mathcal{Z}[T_n]$ is a Laurent series with no positive powers and radius of convergence r not exceeding a by the Cauchy–Hadamard formula

$$r = \lim_{n\to\infty} \|T_n\|^{1/n} \leqslant a .$$

□

Note that if

$$T_n = T^n$$

then $\mathcal{Z}[T_n]$ is up to a factor z identical with the *von Neumann series* of the resolvent of T

$$\mathcal{Z}[T^n] = \sum_{n=0}^{\infty} T^n z^{-n} = \frac{z}{z-T} .$$

Theorem 5 (Inversion formula). The *inverse z-transform* $\mathcal{Z}^{-1}$ is given by

$$\mathcal{Z}_n^{-1}[\mathcal{T}(z)] = \frac{1}{2\pi i} \oint_C z^{n-1} \mathcal{T}(z)\, \mathrm{d}z$$

where C may be any contour enclosing all singularities of $\mathcal{T}(z)$.

Proof. This is just the expression for the coefficients of a Laurent series. □

The following theorems are particularly useful for handling difference equations.

Theorem 6 (Shifting theorem). If $\mathcal{Z}[T_n] = \mathcal{T}(z)$, then for $k \geqslant 0$

$$\mathcal{Z}[T_{n+k}] = z^k \left(\mathcal{T}(z) - \sum_{n=0}^{k-1} T_n z^{-n} \right) .$$

Proof. This follows from

$$\sum_{n=0}^{\infty} T_{n+k} z^{-n} = z^k \sum_{n=0}^{\infty} T_{n+k} z^{-(n+k)}$$

$$= z^k \left(\sum_{n=0}^{\infty} T_n z^{-n} - \sum_{n=0}^{k-1} T_n z^{-n} \right) .$$

□

Theorem 7 (Convolution theorem). Given $\{T_n\}_{n\in\mathbb{N}}$ and $\{T'_n\}_{n\in\mathbb{N}}$ with z-transforms $\mathcal{T}(z)$ and $\mathcal{T}'(z)$ respectively we can define the convolution of the two series by

$$\{T_n * T'_n\}_{n\in\mathbb{N}} = \left\{\sum_{i=0}^{n} T_i T'_{n-i}\right\}_{n\in\mathbb{N}}$$

its z-transform being given by

$$\mathcal{Z}[T_n * T'_n] = \mathcal{T}(z)\mathcal{T}'(z) .$$

Proof. We only need to take into account that $T_n = 0$ for $n < 0$ by definition. Then

$$\mathcal{T}(z)\mathcal{T}'(z) = \sum_{n=0}^{\infty}\sum_{n'=0}^{\infty} T_n T_{n'} z^{-(n+n')} = \sum_{n=0}^{\infty}\sum_{n'=n}^{\infty} T_n T_{n'-n} z^{-n'}$$

$$= \sum_{n'=0}^{\infty}\sum_{n=0}^{n'} T_n T_{n'-n} z^{-n'} = \mathcal{Z}[T_n * T_n'] .$$

□

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